Mean field games on graphs

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Outline

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Introduction

- Mean field games: introduced in 2005 by Pierre-Louis Lions and Jean-Michel Lasry
- Stochastic games when the number of (identical) players $\to \infty$.
- Mean field approach to model interactions between players/agents: each player considers the distribution of others instead of individual states.

Usual framework

- Each individual solves a (stochastic) control problem. The objective function depends on the distribution of other players.
  $\rightarrow$ HJB equation, backward in time.
- The distribution of players evolves according to players' optimal controls.
  $\rightarrow$ transport equation, forward in time.

MFG PDEs (continuous state space - usual framework)

$$(HJB) \qquad \partial_t u + \frac{\sigma^2}{2}\Delta u + H(\nabla u, x, m) = -f(x, m)$$

$$(K) \qquad \partial_t m - \frac{\sigma^2}{2}\Delta m + \nabla \cdot (m \partial_p H(\nabla u, x, m)) = 0$$

$$u(T, x) = g(x, m(T, \cdot)), \qquad m(0, x) = m_0(x)$$

Literature

Research can be divided into 3 strands:

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- Mean field game theory:
    - Existence and uniqueness results (Lasry-Lions, ...)
    - Special cases, mainly linear-quadratic MFGs
    - Long time behavior
    - Limit of N-player games
    - Probabilistic approach (Carmona and Delarue)

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- Mean field game theory:
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  - Special cases, mainly linear-quadratic MFGs
  - Long time behavior
  - Limit of N-player games
  - Probabilistic approach (Carmona and Delarue)
- Numerical approximations of MFG PDEs:
  - General finite difference methods (see e.g. Achdou et al.)
  - Specific schemes with (unusual) monotonocity properties in the case of quadratic Hamiltonians
  - Gradient methods in the case of potential games
  - Talks today

Literature

- Applications:
  - Many toy models (G., Lasry, Lions)
  - Growth models (e.g. Lucas and Moll)
    Remark: Interesting paper "PDE Models in Macroeconomics"
  - Competition between oil producers (Chan, Sircar)
  - Population dynamics (Achdou et al.)
  - Maybe applications to machine learning in the future...

Our goal here is to prove existence and (very general) uniqueness results for mean field games on a graph (discrete state space). In particular, we will consider congestion effects.

Outline

Framework

- Mean Field Games: Continuum of anonymous and identical (indistinguishable) players
- Graph $\mathcal{G}$. Nodes indexed by integers from 1 to N.
- $\forall i \in \mathcal{N} = \{1, \ldots, N\}$:
  - $\mathcal{V}(i) \subset \mathcal{N} \setminus \{i\}$ the set of nodes j for which a directed edge exists from i to j (cardinal: d_i).
  - $\mathcal{V}^{-1}(i) \subset \mathcal{N} \setminus \{i\}$ the set of nodes j for which a directed edge exists from j to i.

Framework (continued)

- Each player's position: Markov chain $(X_t)_t$ with values in $\mathcal{G}$
- Distribution of players: $t \mapsto m(t) = (m(1,t), \dots, m(N,t))$
- Instantaneous transition probabilities at time t:

$$\lambda_t(i,\cdot) : \mathcal{V}(i) \to \mathbb{R}_+ \quad (\forall i \in \mathcal{N})$$

- Running payoff:
  - Running costs: $\mathcal{L}(i, (\lambda_{i,j})_{j \in \mathcal{V}(i)}, m(t))$ to set the value of $\lambda_t(i,j)$ to $\lambda_{i,j}$.
  - Running gain: $f(i, m(t))$, where i is the current state.
- Terminal payoff: $g(i, m(T))$.

Assumptions

- Continuity: $\forall i \in \mathcal{N}$, the functions $\mathcal{L}(i,\cdot,\cdot)$, $f(i,\cdot)$ and $g(i,\cdot)$ are continuous.
- Convexity of the cost functions: $\forall i \in \mathcal{N}, \forall \mu \in \mathbb{R}_+^N, \mathcal{L}(i,\cdot,\mu)$ is a strictly convex function.
- Asymptotic super-linearity of the cost functions:

$$\forall i \in \mathcal{N}, \forall K > 0, \quad \lim_{\lambda \in \mathbb{R}_+^{d_i}, |\lambda| \to +\infty} \inf_{\mu \in [0,K]^N} \frac{\mathcal{L}(i,\lambda,\mu)}{|\lambda|} = +\infty.$$

We define the Hamiltonian functions $\mathcal{H}(i,\cdot,\cdot)$ by:

$$\forall i \in \mathcal{N}, (p,\mu) \in \mathbb{R}^{d_i} \times \mathbb{R}_+^N \mapsto \mathcal{H}(i,p,\mu) = \sup_{\lambda \in \mathbb{R}_+^{d_i}} \lambda \cdot p - \mathcal{L}(i,\lambda,\mu).$$

Nash-MFG equilibrium

- We denote by $\mathcal{A}$ the set of admissible controls λ (L^∞ and deterministic).
- For a given differentiable function $m : t \in [0, T] \mapsto (m(1, t), \dots, m(N, t)) \in \mathcal{P}_N$ (total mass=1) we define:

$$J_m(i, t, \lambda) =$$

$$\mathbb{E}\left[\int_t^T \left(-\mathcal{L}(X_s, \lambda_s(X_s, \cdot), m(s)) + f(X_s, m(s))\right) ds + g\left(X_T, m(T)\right)\right]$$

for $(X_s)_{s\in[t,T]}$ a Markov chain on $\mathcal{G}$, starting from i at time t, with instantaneous transition probabilities given by $(\lambda_s)_{s\in[t,T]}$.

Nash-MFG equilibrium

Nash-MFG equilibrium

A differentiable function $m : t \in [0, T] \mapsto (m(1, t), \dots, m(N, t)) \in \mathcal{P}_N$ is said to be a Nash-MFG equilibrium, if there exists an admissible control $\lambda \in \mathcal{A}$ such that:

$$\forall \tilde{\lambda} \in \mathcal{A}, \forall i \in \mathcal{N}, J_m(i, 0, \lambda) \geq J_m(i, 0, \tilde{\lambda}),$$

and

$$\forall i \in \mathcal{N}, \frac{d}{dt} m(i, t) = \sum_{j \in \mathcal{V}^{-1}(i)} \lambda_t(j, i) m(j, t) - \sum_{j \in \mathcal{V}(i)} \lambda_t(i, j) m(i, t)$$

$\mathcal{G}$-MFG equations

The $\mathcal{G}$-MFG equations consist in a system of $2N$ equations:

$\mathcal{G}$-MFG equations

N Hamilton-Jacobi equations:

$$\frac{d}{dt}u(i,t) + \mathcal{H}\left(i, (u(j,t) - u(i,t))_{j\in\mathcal{V}(i)}, m(1,t), \dots, m(N,t)\right)$$

$$+f(i, m(1,t), \dots, m(N,t)) = 0$$

with $u(i,T) = g(i, m(1,T), \dots, m(N,T))$.
and...

$\mathcal{G}$-MFG equations

$\mathcal{G}$-MFG equations (continued)

... N forward transport equations:

$$\frac{d}{dt}m(i,t) = \sum_{j\in\mathcal{V}^{-1}(i)} \frac{\partial\mathcal{H}(j,\cdot,m(t))}{\partial p_i}\left((u(k,t)-u(j,t))_{k\in\mathcal{V}(j)}\right) m(j,t)$$

$$-\sum_{j\in\mathcal{V}(i)} \frac{\partial\mathcal{H}(i,\cdot,m(t))}{\partial p_j}\left((u(k,t)-u(i,t))_{k\in\mathcal{V}(i)}\right) m(i,t)$$

with $(m(1,0),\ldots,m(N,0)) = m^0$.

Remark: in the literature on MFG, people usually do not care about verification (why?). On graphs, there is no issue as we shall find C^1 solutions.

Outline

Existence

Theorem (Existence)

Under the assumptions made above, there exists a C^1 solution (u, m) of the $\mathcal{G}$-MFG equations.

To prove this result we need a Lemma. This lemma is a comparison principle.

Comparison principle

Lemma (Comparison principle)

Let $m : [0, T] \to \mathcal{P}_N$ *be a continuous function.*
Let $u : [0, T] \to \mathbb{R}^n$ *be a* C^1 *function that verifies:*

$$\forall i, -\frac{d}{dt}u(i,t) - \mathcal{H}\left(i, (u(j,t) - u(i,t))_{j \in \mathcal{V}(i)}, m(t)\right) - f(i, m(t)) \leq 0$$

with $u(i, T) \leq g(i, m(T))$.
Let $v : [0, T] \to \mathbb{R}^n$ *be a* C^1 *function that verifies:*

$$\forall i, -\frac{d}{dt}v(i,t) - \mathcal{H}\left(i, (v(j,t) - v(i,t))_{j \in \mathcal{V}(i)}, m(t)\right) - f(i, m(t)) \geq 0$$

with $v(i, T) \geq g(i, m(T))$.
Then, $\forall i \in \mathcal{N}, \forall t \in [0, T], v(i,t) \geq u(i,t)$.

Sketch of proof for the Theorem

- N HJ equations defines a function $B : m \mapsto u$
- Comparison principle leads a priori bounds on u

$$\sup_{i \in \mathcal{N}} \|u(i,\cdot)\|_\infty \leq \sup_{i \in \mathcal{N}} \|g(i,\cdot)\|_\infty$$

$$+T \sup_{i \in \mathcal{N}, \mu \in \mathcal{P}_N} |\mathcal{H}(i, 0, \mu)| + T \sup_{i \in \mathcal{N}, \mu \in \mathcal{P}_N} |f(i, \mu)|.$$

- N transport equations defines a function $F : u \mapsto m$
- Bounds on u lead to bounds on $\frac{dm}{dt}$
- Ascoli + Schauder permit to obtain a fixed point to $m \mapsto F(B(m))$.

Outline

Classical uniqueness result

It is possible to derive a uniqueness result equivalent to the one presented in Lasry-Lions' initial paper:

Theorem (Classical Uniqueness)

Assume that g is such that:

$$\forall(\nu,\mu) \in \mathcal{P}_N \times \mathcal{P}_N, \sum_{i=1}^{N}(g(i,\nu) - g(i,\mu))(\nu_i - \mu_i) \geq 0 \implies \nu = \mu.$$

Assume that f is such that:

$$\forall(\nu,\mu) \in \mathcal{P}_N \times \mathcal{P}_N, \sum_{i=1}^{N}(f(i,\nu) - f(i,\mu))(\nu_i - \mu_i) \geq 0 \implies \nu = \mu$$

Classical uniqueness result

Theorem (Classical Uniqueness (continued))

*Assume that $\mathcal{H}$ **does not** depend on m.*
Then, if $(\widehat{u}, \widehat{m})$ and $(\tilde{u}, \tilde{m})$ are two C^1 solutions of the $\mathcal{G}$-MFG equations, we have $\widehat{m} = \tilde{m}$ and $\widehat{u} = \tilde{u}$.

*In words, this says that if people **do not** want to live together and if the cost to move **does not** depend on the number of people at specific nodes, then uniqueness is ensured.*
How can this be extended? By adding m in the the Hamiltonian functions!
$\rightarrow$ Congestion models for instance.

General uniqueness result

Theorem (Uniqueness)

Assume that g is such that:

$$\forall(\nu,\mu) \in \mathcal{P}_N \times \mathcal{P}_N, \sum_{i=1}^{N}(g(i,\nu) - g(i,\mu))(\nu_i - \mu_i) \geq 0 \implies \nu = \mu.$$

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Assume that $\mathcal{H}$ is such that $\forall i \in \mathcal{N}, \forall j \in \mathcal{V}(i), \frac{\partial \mathcal{H}}{\partial p_j}(i,\cdot,\cdot)$ is a C^1 function on $\mathbb{R}^{d_i} \times \mathbb{R}_+^N$.

General uniqueness result

Theorem (Uniqueness (continued))

Let us define
$A : (q_1, \ldots, q_N, \mu) \in \prod_{i=1}^N \mathbb{R}^{d_i} \times \mathcal{P}_N \mapsto (\alpha_{ij}(q_i, \mu))_{i,j} \in \mathcal{M}_N$ *by:*

$$\alpha_{ij}(q_i, \mu) = -\frac{\partial \mathcal{H}}{\partial \mu_j}(i, q_i, \mu).$$

Let us also define
$\forall i \in \mathcal{N},\ B^i : (q_i, \mu) \in \mathbb{R}^{d_i} \times \mathcal{P}_N \mapsto \left(\beta^i_{jk}(q_i, \mu)\right)_{j,k} \in \mathcal{M}_{N,d_i}$ *by:*

$$\beta^i_{jk}(q_i, \mu) = \frac{\mu_i}{2} \frac{\partial^2 \mathcal{H}}{\partial \mu_j \partial q_{ik}}(i, q_i, \mu).$$

General uniqueness result

Theorem (Uniqueness (continued))

Let us also define, $\forall i \in \mathcal{N}$,
$C^i : (q_i, \mu) \in \mathbb{R}^{d_i} \times \mathcal{P}_N \mapsto \left(\gamma^i_{jk}(q_i, \mu)\right)_{j,k} \in \mathcal{M}_{d_i,N}$ *defined by:*

$$\gamma^i_{jk}(q_i, \mu) = \frac{\mu_i}{2} \frac{\partial^2 \mathcal{H}}{\partial \mu_k \partial q_{ij}}(i, q_i, \mu).$$

Let us finally define, $\forall i \in \mathcal{N}$,
$D^i : (q_i, \mu) \in \mathbb{R}^{d_i} \times \mathcal{P}_N \mapsto \left(\delta^i_{jk}(q_i, \mu)\right)_{j,k} \in \mathcal{M}_{d_i}$ *defined by:*

$$\delta^i_{jk}(q_i, \mu) = \mu_i \frac{\partial^2 \mathcal{H}}{\partial q_{ij} \partial q_{ik}}(i, q_i, \mu).$$

General uniqueness result

Theorem (Uniqueness (continued))

Assume that $\forall (q_1, \dots, q_N, \mu) \in \prod_{i=1}^N \mathbb{R}^{d_i} \times \mathcal{P}_N$:

$$\begin{pmatrix} A(q_1,\dots,q_N,\mu) & B^1(q_1,\mu) & \cdots & \cdots & \cdots & B^N(q_N,\mu) \\ C^1(q_1,\mu) & D^1(q_1,\mu) & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & 0 \\ C^N(q_N,\mu) & 0 & \cdots & \cdots & 0 & D^N(q_N,\mu) \end{pmatrix} \geq 0.$$

Then, if $(\widehat{u}, \widehat{m})$ *and* $(\tilde{u}, \tilde{m})$ *are two* C^1 *solutions of the* $\mathcal{G}$*-MFG equations, we have* $\widehat{m} = \tilde{m}$ *and* $\widehat{u} = \tilde{u}$.

General uniqueness result – Proof

Sketch of proof:

Classical idea of computing the value of

$$I = \sum_{i=1}^{N} \int_0^T \frac{d}{dt} \left((\widehat{u}(i,t) - \tilde{u}(i,t))(\widehat{m}(i,t) - \tilde{m}(i,t)) \right) dt$$

in two different ways.

We first know directly that

$$I = \sum_{i=1}^{N} (g(i, \widehat{m}(T)) - g(i, \tilde{m}(T)))(\widehat{m}(i,T) - \tilde{m}(i,T)).$$

General uniqueness result – Proof

We introduce:

- $M(q_1, \ldots, q_N, \mu)$ the matrix of the Theorem
- Convex combinations of solutions: $u^\theta = \theta \widehat{u} + (1-\theta)\tilde{u}$ and $m^\theta = \theta \widehat{m} + (1-\theta)\tilde{m}$
- A vector V:

$$V(t) = (\widehat{m}(t) - \tilde{m}(t), ((\widehat{u}(k,t) - \tilde{u}(k,t)) - (\widehat{u}(1,t) - \tilde{u}(1,t)))_{k \in \mathcal{V}(1)}, \ldots, ((\widehat{u}(k,t) - \tilde{u}(k,t)) - (\widehat{u}(N,t) - \tilde{u}(N,t)))_{k \in \mathcal{V}(N)}).$$

General uniqueness result – Proof

If we differentiate the expression in I and rearrange the terms (not obvious), we have:

$$I = -\int_0^T \sum_{i=1}^N (f(i,\widehat{m}(t)) - f(i,\tilde{m}(t)))(\widehat{m}(i,t) - \tilde{m}(i,t))dt + J,$$

where

$$J = \int_0^T \int_0^1 V(t) M((u^\theta(k,t) - u^\theta(1,t))_{k\in\mathcal{V}(1)},$$

$$\ldots, (u^\theta(k,t) - u^\theta(N,t))_{k\in\mathcal{V}(N)}, m^\theta(t)) V(t)' d\theta dt.$$

Uniqueness then comes from $M \geq 0$ and f and g decreasing.

Outline

Potential games

We consider the special case where:

- The Hamiltonian functions (or equivalently the cost functions) do not depend on m.
- There are two functions F and G such that: $\frac{\partial F}{\partial m_i} = f(i, \cdot)$ and $\frac{\partial G}{\partial m_i} = g(i, \cdot)$

Potential games

We consider the problem of **one** planner:

$$\sup_{\lambda \in \mathcal{A}} \mathcal{J}(0, m^0, \lambda)$$

where $\mathcal{J}(t, m^t, \lambda)$

$$= \int_t^T \left(F(m(s)) - \sum_{i=1}^{N} L(i, (\lambda_s(i,j))_{j \in \mathcal{V}(i)}) m(s,i) \right) ds + G(m(T))$$

where $\forall i \in \mathcal{N}, m(t,i) = m_i^t$ and $\forall i \in \mathcal{N}, \forall s \in [t, T]$,

$$\frac{d}{ds} m(s,i) = \sum_{j \in \mathcal{V}^{-1}(i)} \lambda_s(j,i) m(s,j) - \sum_{j \in \mathcal{V}(i)} \lambda_s(i,j) m(s,i).$$

Potential games - HJ equation

The HJ equation associated to this problem is:

$$\frac{\partial \Phi}{\partial t}(t, m_1, \ldots, m_N) + \mathcal{H}(m_1, \ldots, m_N, \nabla\Phi) + F(m_1, \ldots, m_N) = 0$$

Terminal conditions: $\Phi(T, m_1, \ldots, m_N) = G(m_1, \ldots, m_N)$.
The Hamiltonian function is $\mathcal{H}(m_1, \ldots, m_N, p) =$

$$\sup_{(\lambda_{i,j})_{i\in\mathcal{N},j}} \sum_{i=1}^{N} \left[\left(\sum_{j\in\mathcal{V}^{-1}(i)} \lambda_{j,i} m_j - \sum_{j\in\mathcal{V}(i)} \lambda_{i,j} m_i \right) p_i - L(i, (\lambda_{i,j})_{j\in\mathcal{V}(i)}) m_i \right]$$

$$= \sum_{i=1}^{N} m_i H\left(i, (p_j - p_i)_{j\in\mathcal{V}(i)}\right)$$

The Master equation

Differenciating formally we obtain for $U = \nabla \Phi$ the following equations:

$$0 = \frac{\partial U_i}{\partial t} + H\left(i, (U_j - U_i)_{j \in \mathcal{V}(i)}\right)$$

$$+ \sum_{j=1}^{N} m_j \sum_{k \in \mathcal{V}(j)} \left(\frac{\partial U_i}{\partial m_k} - \frac{\partial U_i}{\partial m_j} \right) \frac{\partial H(j, \cdot)}{\partial p_k} \left((U_l - U_j)_{l \in \mathcal{V}(j)} \right) + f(i, \cdot)$$

$$U_i(T, m) = g(i, m)$$

This would be the HJ equation of the initial MFG equation, had we considered m a state variable. This is the **Master equation**!

Conclusion

With potential games:

- 1 HJ equations
- N equations (Master equation)
- $2N$ equations for the underlying MFG problem. A solution is:

$$m(0) = m^0, \qquad \frac{d}{dt} m(t,i) =$$

$$\sum_{j \in \mathcal{V}^{-1}(i)} \frac{\partial H(j,\cdot)}{\partial p_i} \left((U_k(t,m(t)) - U_j(t,m(t)))_{k \in \mathcal{V}(j)} \right) m(t,j)$$

$$- \sum_{j \in \mathcal{V}(i)} \frac{\partial H(i,\cdot)}{\partial p_j} \left((U_k(t,m(t)) - U_i(t,m(t)))_{k \in \mathcal{V}(i)} \right) m(t,i)$$

$$\forall i \in \mathcal{N}, u(t,i) = U_i(t, m(t,1), \ldots, m(t,N))$$

Questions?